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Topological defects in spinor condensates

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Abstract
We investigate the structure of topological defects in the ground states of spinor Bose–Einstein condensates with spin $F = 1$ or $F = 2$. The type and number of defects are determined by calculating the first and second homotopy groups of the order-parameter space. The order-parameter space is identified with a set of degenerate ground state spinors. Because the structure of the ground state depends on whether or not there is an external magnetic field applied to the system, defects are sensitive to the magnetic field. We study both cases and find that the defects in zero and non-zero field are different.

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1. Introduction

Bose–Einstein condensates (BECs) of alkali atoms have an internal degree of freedom due to the hyperfine spin of these atoms. If a BEC is realized in a magnetic trap this degree of freedom is frozen and in a mean-field limit the condensate is described by a scalar order-parameter. However, if an optical trap [1] is used to confine condensate atoms, this degree of freedom is liberated and has to be taken into account [2, 3]. Condensates with this property are called spinor or vector condensates. In the mean-field theory the ground state of a spinor condensate is described by an order-parameter $\Psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}\xi(\mathbf{r})$, where $n(\mathbf{r})$ is the density of the condensate and $\xi(\mathbf{r})$ is a normalized spinor, $\xi^\dagger(\mathbf{r})\xi(\mathbf{r}) = 1$. In this paper, the density n is assumed to be constant. Because of the vectorial nature of the order-parameter, the behaviour of spinor condensates is in many ways different from that of scalar condensates. One manifestation of this can be seen in the difference of defects in scalar and spinor condensates. In the former vortices with integer winding numbers can exist. The latter allow for more complex defects, which are the topic of this paper. Our study is based on the ground states calculated using the mean-field theory and single condensate approximation [2, 7–9]. Mean-field theory is widely used in the study of Bose–Einstein condensates and

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it is usually assumed to give a good description of the physical system. However, some results suggest that the actual ground states of spinor condensates may be different from those obtained using mean-field theory [18, 19]. Thus, the results of this paper are valid only as long as mean-field theory can be applied.

2. Characterization of the used techniques

2.1. The order-parameter space

In condensed matter systems the concept of an order-parameter is very important [4, 5]. Order-parameter $f(\mathbf{r})$ is a continuous mapping from some region of the physical space into the order-parameter space M, which consists of all possible values of the order-parameter. It is usually possible to associate the order-parameter space with a group G that acts on that space. If this action is transitive (i.e., for every $x, x' \in M$ there exists some $g \in G$ for which $x' = g \cdot x$), we can arbitrarily choose some element $x_{\mathrm{ref}} \in M$ which we call the reference order-parameter. Every element $x \in M$ can then be obtained from x_{ref} by acting on it by a suitable element of the group G. Those elements of G which leave x_{ref} fixed constitute a subgroup H called the isotropy group. Explicitly $H = \{g \in G | g \cdot x_{\mathrm{ref}} = x_{\mathrm{ref}}\}$. Under some rather general requirements for G and M the order-parameter space M can be identified with the quotient space G/H. When considering the defects of spin-F Bose–Einstein condensate the order-parameter is the normalized $(2F+1)$-component spinor $\xi(\mathbf{r}) \in \mathbb{C}^{2F+1}$. Because we study what kind of defects can exist in the ground state of the system, the order-parameter space is the set of spinors that minimize the energy. In the absence of an external magnetic field we can often choose $G = U(1) \times SO(3)$. However, this choice is not always the correct one, as there may be order-parameter spaces in which $U(1) \times SO(3)$ does not act transitively; see below. $U(1) \times SO(3)$ acts on $\mathbb{C}^{2F+1}$ via equation $((c, R), \xi) \mapsto cD^{(F)}(R)\xi$, where $(c, R) \in U(1) \times SO(3)$ and $D^{(F)}$ is the $(2F+1)$-dimensional irreducible representation of $SO(3)$. This representation is given by the map $R(\alpha, \beta, \gamma) \mapsto D^{(S)}(\alpha, \beta, \gamma)$, where $R(\alpha, \beta, \gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma) \in SO(3)$ is given as a product of rotations about y- and z-axes and $D^{(S)}(\alpha, \beta, \gamma) = \exp(-\mathrm{i}\alpha F_z)\exp(-\mathrm{i}\beta F_y)\exp(-\mathrm{i}\gamma F_z)$. Here F_y and F_z are y- and z-components of the spin matrices corresponding to spin F. Representation matrices for $F = 1$ and $F = 2$ are given in the appendix.

In the presence of an external magnetic field the energy of the ground state is invariant under gauge transformations and rotations about the axis of the magnetic field, so we choose $G = U(1) \times SO(2)$.

2.2. Homotopy groups

Homotopy groups of the order-parameter space describe physical defects [4]. The nth homotopy group $\pi_n(M)$ of the space M consists of the equivalence classes of continuous maps from n-dimensional sphere S^n to the space M. Two maps are equivalent if they are homotopic to one another. In physics, the first and second homotopy groups are of special importance. The first homotopy group $\pi_1(M)$ describes singular line defects and domain walls, which are non-singular defects. The second homotopy group $\pi_2(M)$ describes singular point defects and non-singular line defects. Thus, identifying M with G/H, we can learn much from the possible defects in a physical system if we know $\pi_1(G/H)$ and $\pi_2(G/H)$. These can be calculated with the help of the following theorem.

Table 1. The reference spinors and their general forms for $F = 1$ spinor condensate when the external magnetic field is zero.

	ξ_{ref}^T	$\xi(\alpha,\beta,\gamma,\theta)^T$
f	$(1, 0, 0)$	$\mathrm{e}^{\mathrm{i}(\theta-\gamma)}\left(\mathrm{e}^{-\mathrm{i}\alpha}\cos^2\frac{\beta}{2}, \frac{1}{\sqrt{2}}\sin\beta, \mathrm{e}^{\mathrm{i}\alpha}\sin^2\frac{\beta}{2}\right)$
af	$(0, 1, 0)$	$\mathrm{e}^{\mathrm{i}\theta}\left(-\mathrm{e}^{-\mathrm{i}\alpha}\frac{1}{\sqrt{2}}\sin\beta, \cos\beta, \mathrm{e}^{\mathrm{i}\alpha}\frac{1}{\sqrt{2}}\sin\beta\right)$

Theorem 1. *Let G be a Lie group with $\pi_0(G) = \pi_1(G) = \pi_2(G) = 0$. Here 0 denotes a one-element group. Let $H \subseteq G$ be a closed subgroup, and $H_0 \subseteq H$ the connected component of the identity. There are isomorphisms*

$$\pi_1(G/H) \cong H/H_0 \tag{1}$$

and

$$\pi_2(G/H) \cong \pi_1(H_0). \tag{2}$$

For a proof, see [4] or [15].

We cannot use this theorem if G is $U(1) \times SO(3)$, because $\pi_1(U(1) \times SO(3)) = \mathbb{Z} \times \mathbb{Z}_2$. This problem can be solved by using $\mathbb{R} \times SU(2)$ instead of $U(1) \times SO(3)$, since this group fulfils the requirements of the theorem. The former is a covering group of the latter, the covering projection $P : \mathbb{R} \times SU(2) \to U(1) \times SO(3)$ being given by $(x, \mathcal{U}(\alpha,\beta,\gamma)) \mapsto (\mathrm{e}^{\mathrm{i}x}, R(\alpha,\beta,\gamma))$, where $x \in \mathbb{R}$ and

$$\mathcal{U}(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)/2} & -\sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}(\gamma-\alpha)/2} \\ \sin\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}(\gamma-\alpha)/2} & \cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}(\alpha+\gamma)/2} \end{pmatrix} \in SU(2) \tag{3}$$

Every matrix in $SU(2)$ can be written in this form. Sufficient intervals for α, β and γ are $[0, 2\pi]$, $[0, \pi]$ and $[0, 4\pi]$, respectively.

3. Spin 1

The ground state structure for $F = 1$ condensate was calculated by Ho [2] and by Ohmi and Machida [3]. If the external magnetic field is non-zero the spin-dependent part in the energy is $\mathcal{E}(\xi) = c\langle\mathbf{F}\rangle_\xi^2 - p\langle F_z\rangle_\xi + q\langle F_z^2\rangle_\xi$ [11]. Here the kinetic energy term is neglected in the Thomas–Fermi approximation and $\langle\mathbf{F}\rangle_\xi = \xi^\dagger\mathbf{F}\xi$. The constant c depends on scattering lengths and the density n, whereas p describes linear and q quadratic Zeeman interaction with the external magnetic field. The external field is assumed to be directed along the z-axis.

3.1. Zero external field

The equation for energy is obtained by setting $p = q = 0$. Depending on the sign of c energy is minimized either by $\langle\mathbf{F}\rangle_\xi^2 = 1$ or $\langle\mathbf{F}\rangle_\xi = 0$. The former is called the ferromagnetic (f) and the latter the antiferromagnetic (af) phase. The order-parameter spaces corresponding to these phases are $M^f = \left\{\xi \in \mathbb{C}^3 | \langle\mathbf{F}\rangle_\xi^2 = 1, \xi^\dagger\xi = 1\right\}$ and $M^{af} = \{\xi \in \mathbb{C}^3 | \langle\mathbf{F}\rangle_\xi = 0, \xi^\dagger\xi = 1\}$. It is easy to see that $U(1) \times SO(3)$ acts transitively on these sets. The reference order-parameters and general order-parameters obtained from these by a rotation and gauge transformation are shown in table 1.

3.1.1. Ferromagnetic phase. From table 1 we see that we do not need the angle θ because γ can produce all possible gauge transformations. This means that instead of $\mathbb{R} \times SU(2)$ we can use only $SU(2)$. To find the elements of the isotropy group H^f we set $\xi^f(\alpha, \beta, \gamma, 0) = \xi^f_{\text{ref}}$. This gives $H^f = \{\mathbb{I}, -\mathbb{I}\}$. Using theorem 1 we get $\pi_1(G/H^f) \cong \{\pm\mathbb{I}\}$ and $\pi_2(G/H^f) \cong 0$. The order-parameter space is $SU(2)/\{\pm\mathbb{I}\} \cong SO(3)$. This order-parameter space has also been encountered in ^{3}He–A [12].

Physically, these results mean that we can have one non-trivial singular vortex but we cannot have any non-trivial monopoles, i.e. singular point defects. The non-trivial vortex is a defect in which the overall phase of the spinor changes by 2π as the defect line is encircled.

3.1.2. Antiferromagnetic phase. Now the isotropy group is $H^{af} = \{(n2\pi, a(\varphi)), ((n + \frac{1}{2})2\pi, ga(\varphi)) | \varphi \in [0, 4\pi], n \in \mathbb{Z}\}$, where we have defined $a(\varphi) = \mathcal{U}(\varphi, 0, 0)$ and $g = \mathcal{U}(0, \pi, 0)$. The connected component of the identity is $H^{af}_0 = \{(0, a(\varphi)) \mid \varphi \in [0, 4\pi]\}$. We get $\pi_1(G/H^{af}) \cong \{(n2\pi, \mathbb{I})H^{af}_0, ((n + \frac{1}{2})2\pi, g)H^{af}_0 | n \in \mathbb{Z}\}$. This group is isomorphic to $\mathbb{Z}$, the isomorphism being given by the map $((2n + j)\pi, g^j)H^{af}_0 \mapsto 2n + j$, where $j = 0$ or 1 and $g^0 \equiv \mathbb{I}$. Thus $\pi_1(G/H^{af}) \cong \mathbb{Z}$. Previously, the order-parameter space and first homotopy group were concluded to be $U(1) \times S^2$ and $\mathbb{Z}$ [2] or $[U(1) \times S^2]/\mathbb{Z}_2$ and $\mathbb{Z} \times \mathbb{Z}_2$ [6]. Both of these order-parameter spaces are incorrect but the first homotopy group in [2] is correct. However, that of [6] is not correct, since there is no (group) isomorphism between $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}_2$. The isotropy group (in $U(1) \times SO(3)$) is isomorphic to $O(2)$, but it cannot be expressed as a direct product of a subgroup of $U(1)$ and $SO(3)$. Thus the order-parameter space can be written only as $G/H = [U(1) \times SO(3)]/O(2)_{G+S}$, where $G + S$ means that the isotropy group consists of gauge transformations performed simultaneously with spin rotations.

Because H^{af}_0 is homeomorphic to $U(1)$ and homeomorphic spaces have the same homotopy groups, we get $\pi_2(G/H^{af}) \cong \pi_2(U(1)) \cong \mathbb{Z}$. If we move around a closed path in the condensate we note that when we return to the starting point the angle θ has changed by some amount. If we define the change in this angle divided by 2π to be the winding number, we see from the elements of H^{af}/H^{af}_0 that the winding number can be either an integer (n) or a half-integer $(\frac{1}{2} + n)$ [10]. Paths in the order-parameter space can be represented pictorially as follows. From table 1 we see that we have three parameters in the general expression for the ferromagnetic state. From these α and β can be restricted to the intervals $[0, 2\pi]$ and $[0, \pi]$, respectively, and $\theta \in [0, 2\pi]$. However, because $\xi^{af}(\alpha \pm \pi, \pi - \beta, \gamma, \theta) = -\xi^{af}(\alpha, \beta, \gamma, \theta)$, we can actually restrict θ to the interval $[0, \pi]$. These parameters can be represented using cylindrical coordinates (α, r, z), where now $r = \beta$, $z = \theta$, see figure 1.

In summary, possible line defects are those in which the overall phase changes by $2\pi n$ as the defect line is encircled and those in which a phase change of $\pi + 2\pi n$ is accompanied by a 180° spinor rotation. Also point defects, labelled by integers, are possible.

3.2. Non-zero external field

Straightforward minimization of energy gives four ground states which are degenerate with respect to one or two phase variables, see table 2. The identification of the order-parameter space G/H is easier than in the absence of the magnetic field.

In f_1, f_2 and af states $G = U(1)$ and $H = 1$, so $G/H = U(1)$ for which $\pi_1(U(1)) \cong \mathbb{Z}$ and $\pi_2(U(1)) \cong 0$. Thus, we can have singular vortices with arbitrary integer winding numbers but we do not have singular point defects. This resembles the situation in a scalar condensate, where we have similar defects.

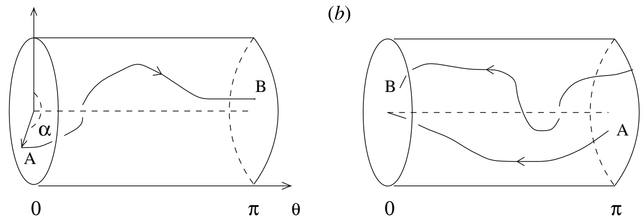

Figure 1. Pictorial representation of the parameters of the antiferromagnetic spinor. Angles α and θ are shown in the picture and angle β is the length of the vector. On the boundary points $(\alpha, \beta, 0)$ and $(\alpha \pm \pi, \pi - \beta, \pi)$ correspond to the same value of the order-parameter. This is also true for all points for which $\beta = \pi$, θ is fixed and $\alpha \in [0, 2\pi)$. A (B) denotes the starting (ending) point of the curve. We assume $\xi(A) = \xi(B)$, so the curves define closed curves in the order-parameter space. Defining the direction of increasing θ to be positive, we see in (*a*) a path with winding number $\frac{1}{2}$ and in (*b*) a path with winding number -1.

Table 2. The ground states for $F = 1$ spinor condensate when the external magnetic field is non-zero. States are degenerate with respect to angles θ and ϕ. We have assumed that $c \neq 0$. f_3 evolves to f_1 (f_2) state as p reaches $2c$ $(-2c)$.

	ξ_{ref}^T	$\mathcal{E}(\xi)$
f_1	$\mathrm{e}^{\mathrm{i}\theta}(1, 0, 0)$	$c - p + q$
f_2	$\mathrm{e}^{\mathrm{i}\theta}(0, 0, 1)$	$c + p + q$
f_3	$\left(\mathrm{e}^{\mathrm{i}\theta}\sqrt{\frac{1}{2} + \frac{p}{4c}}, 0, \mathrm{e}^{\mathrm{i}\phi}\sqrt{\frac{1}{2} - \frac{p}{4c}}\right)$	$-\frac{p^2}{4c} + q$
af	$\mathrm{e}^{\mathrm{i}\theta}(0, 1, 0)$	0

In the f_3 state $G/H = U(1) \times U(1)$ for which $\pi_1(U(1) \times U(1)) \cong \mathbb{Z} \times \mathbb{Z}$ and $\pi_2(U(1) \times U(1)) \cong 0$. Now we can have independent vortices in the $m = 1$ and $m = -1$ components of the spinor.

4. Spin 2

The ground states for $F = 2$ spinor condensate were calculated by Ciobanu *et al* [7] and Ueda and Koashi [8]. In the Thomas–Fermi approximation the spin-dependent energy is given by $\mathcal{E}(\xi) = c\langle \mathbf{F}\rangle_\xi^2 + d|\Theta_\xi|^2 - p\langle F_z\rangle_\xi$, where c and d are constants depending on scattering lengths and p describes the linear Zeeman effect. The possible ground states are characterized by two parameters, namely $\|\langle \mathbf{F}\rangle_\xi\| = \left(\langle \mathbf{F}\rangle_\xi^2\right)^{1/2}$ and $|\Theta_\xi| = \left|2\xi_2\xi_{-2} - 2\xi_1\xi_{-1} + \xi_0^2\right|$.

4.1. Zero external field

The energy in zero magnetic field is obtained by setting $p = 0$. Because $\langle \mathbf{F}\rangle_\xi^2$ and $|\Theta_\xi|$ are invariant under the action of $U(1) \times SO(3)$, and for each ξ there exists a rotation R for which $\langle \mathbf{F}\rangle_\xi^2 = \langle F_z\rangle_{D(R)\xi}^2$, we can write the energy in the form $\mathcal{E}(\xi) = c\langle F_z\rangle_\xi^2 + d|\Theta_\xi|^2$. This equation has been solved in [7, 8]. It turns out that there are three possible phases in the system, ferromagnetic (F, F'), cyclic (C) and polar (P).

To express the order-parameter spaces we define $M(i, j) = \{\xi \in \mathbb{C}^5 | \|\langle \mathbf{F}\rangle_\xi\| = i, |\theta_\xi| = j, \xi^\dagger\xi = 1\}$. Then in F phase the order-parameter space is $M(2, 0)$, in F' phase $M(1, 0)$, in C phase $M(0, 0)$ and in P phase $M(0, 1)$. Representative spinors from these sets and their energies are shown in table 3. It turns out that $U(1) \times SO(3)$ acts transitively on the

Table 3. Ground state spinors and their energies of $F = 2$ condensate when the external magnetic field is absent. General forms of the ground states can be obtained from these by a rotation and a gauge transformation. Note that in the P state we have two free parameters. These are needed because we cannot obtain every possible spinor representing the polar state from a fixed reference spinor by a rotation and a gauge transformation.

	ξ^T	$\mathcal{E}$
F	$(1, 0, 0, 0, 0)$	$4c$
F'	$(0, 1, 0, 0, 0)$	c
C	$\frac{1}{2}(1, 0, \sqrt{2}, 0, -1)$	0
P	$\frac{1}{\sqrt{2}}(\sin\phi\sin\psi, \sin\phi\cos\psi, \sqrt{2}\cos\phi, -\sin\phi\cos\psi, \sin\phi\sin\psi)$	d

parameter spaces of the ferromagnetic and cyclic phases. However, this is not true for the order-parameter space of the polar phase. For example, if we first choose $\phi = 0$ and then $\phi = \frac{\pi}{2}, \psi = 0$ in the spinor representing a polar state, we get two spinors which cannot be converted to each other by a rotation and gauge transformation. This means $U(1) \times SO(3)$ is not a group large enough in the case of polar phase.

4.1.1. Ferromagnetic phases. There are two possible ferromagnetic phases, labelled by F and F'. As in $F = 1$ case, in both of these phases we can use $SU(2)$ instead of $\mathbb{R} \times SU(2)$.

In F phase $H^F = H^F \big/ H_0^F = \{\mathbb{I}, (-\mathrm{i}\sigma_z), (-\mathrm{i}\sigma_z)^2, (-\mathrm{i}\sigma_z)^3\}$ and thus $\pi_1(G/H^F) \cong \mathbb{Z}_4$, $\pi_2(G/H^F) \cong 0$. Here σ_z is the z-component of Pauli matrices and $-\mathrm{i}\sigma_z$ describes rotation about the z-axis through 180°. Non-trivial vortices are those in which the reference spinor rotates through 180°, 360° or 540° about the z-axis when the defect line is circulated. In $SO(3)$ the isotropy group is $\{\pm\mathbb{I}\}$, so $G/H = SO(3)/\mathbb{Z}_2$. For a pictorial representation of paths in $SO(3)/\mathbb{Z}_2$ see [13].

In F' phase the order-parameter space is $SO(3)$, and defects are similar to those in the ferromagnetic phase of spin-1 condensate.

4.1.2. Cyclic phase. In C phase we meet an example of a non-commuting first homotopy group. A rotation and a gauge transformation of the reference spinor give

$$\xi^C = \frac{1}{2}\,\mathrm{e}^{\mathrm{i}\theta}\begin{pmatrix} \mathrm{e}^{-2\mathrm{i}\alpha}\left(\cos^4\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}2\gamma} + \frac{\sqrt{3}}{2}\sin^2\beta - \sin^4\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}2\gamma}\right) \\ \mathrm{e}^{-\mathrm{i}\alpha}\sin\beta\left(\cos^2\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}2\gamma} - \frac{\sqrt{3}}{2}\sin 2\beta + \sin^2\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}2\gamma}\right) \\ -\mathrm{i}\frac{\sqrt{6}}{2}\sin^2\beta\sin 2\gamma + \frac{\sqrt{2}}{4}(1 + 3\cos 2\beta) \\ \mathrm{e}^{\mathrm{i}\alpha}\sin\beta\left(\sin^2\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}2\gamma} + \frac{\sqrt{3}}{2}\sin 2\beta + \cos^2\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}2\gamma}\right) \\ \mathrm{e}^{2\mathrm{i}\alpha}\left(\sin^4\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}2\gamma} + \frac{\sqrt{3}}{2}\sin^2\beta - \cos^4\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}2\gamma}\right) \end{pmatrix}.$$

Equating this with $\frac{1}{2}(1, 0, \sqrt{2}, 0, -1)^T$ yields the elements of the isotropy group. H^C turns out to be a discrete, non-commuting group. Explicitly H^C is the union of the conjugacy classes shown below. The isotropy group (in $U(1) \times SO(3)$) is isomorphic to the tetrahedral group, which is the symmetry group of a tetrahedron. H^C is a discrete group and thus $\pi_1(G/H^C) = H^C$. Because H^C is a non-commuting group we have to use the conjugacy classes of $\pi_1(G/H^C)$ to classify the topologically inequivalent defects [4]. Two line defects are topologically equivalent if and only if they are characterized by the same conjugacy class of the first homotopy group. Defects can still be labelled by the elements of the first homotopy group, but if these elements belong to the same conjugacy class, corresponding defects can

Table 4. The multiplication table of the conjugacy classes of C phase. Because the class multiplication is commutative only half of that is shown. Winding numbers have been omitted for clarity. When two classes are multiplied the winding number of the resulting class is the sum of the individual winding numbers.

	$\overline{C_0}$	C_2	C_3	$\overline{C_3}$	C_3^2	$\overline{C_3^2}$
$\overline{C_0}$	C_0					
C_2	C_2	$6C_0+6\overline{C_0}+4C_2$				
C_3	$\overline{C_3}$	$3(C_3+\overline{C_3})$	$3\overline{C_3^2}+C_3^2$			
$\overline{C_3}$	C_3	$3(C_3+\overline{C_3})$	$\overline{C_3^2}+3C_3^2$	$3\overline{C_3^2}+C_3^2$		
C_3^2	$\overline{C_3^2}$	$3\left(C_3^2+\overline{C_3^2}\right)$	$4\overline{C_0}+2C_2$	$4C_0+2C_2$	$3C_3+\overline{C_3}$	
$\overline{C_3^2}$	C_3^2	$3\left(C_3^2+\overline{C_3^2}\right)$	$4C_0+2C_2$	$4\overline{C_0}+2C_2$	$C_3+3\overline{C_3}$	$3C_3+\overline{C_3}$

be continuously transformed to one another. However, if they belong to different conjugacy classes this is not possible. The conjugacy classes are

$$\begin{aligned}
&C_0(n)=\{(n,\mathbb{I})\},\quad \overline{C_0}(n)=\{(n,-\mathbb{I})\}\\
&C_2(n)=\{(n,a),(n,-a),(n,b),(n,-b),(n,c),(n,-c)\}\\
&C_3\left(\tfrac{1}{3}+n\right)=\left\{\left(\tfrac{1}{3}+n,d\right),\left(\tfrac{1}{3}+n,e\right),\left(\tfrac{1}{3}+n,f\right),\left(\tfrac{1}{3}+n,g\right)\right\}\\
&\overline{C_3}\left(\tfrac{1}{3}+n\right)=\left\{\left(\tfrac{1}{3}+n,-d\right),\left(\tfrac{1}{3}+n,-e\right),\left(\tfrac{1}{3}+n,-f\right),\left(\tfrac{1}{3}+n,-g\right)\right\}\\
&C_3^2\left(\tfrac{2}{3}+n\right)=\left\{\left(\tfrac{2}{3}+n,d^2\right),\left(\tfrac{2}{3}+n,e^2\right),\left(\tfrac{2}{3}+n,f^2\right),\left(\tfrac{2}{3}+n,g^2\right)\right\}\\
&\overline{C_3^2}\left(\tfrac{2}{3}+n\right)=\left\{\left(\tfrac{2}{3}+n,-d^2\right),\left(\tfrac{2}{3}+n,-e^2\right),\left(\tfrac{2}{3}+n,-f^2\right),\left(\tfrac{2}{3}+n,-g^2\right)\right\}.
\end{aligned}\tag{4}$$

Here $n\in\mathbb{Z}$, $a=\mathcal{U}(\pi,0,0)$, $b=\mathcal{U}\left(0,\pi,\frac{\pi}{2}\right)$, $c=\mathcal{U}\left(0,\pi,\frac{3\pi}{2}\right)$, $d=\mathcal{U}\left(\frac{\pi}{4},\frac{\pi}{2},\frac{\pi}{4}\right)$, $e=\mathcal{U}\left(\frac{\pi}{4},\frac{\pi}{2},\frac{13\pi}{4}\right)$, $f=\mathcal{U}\left(\frac{13\pi}{4},\frac{\pi}{2},\frac{\pi}{4}\right)$, $g=\mathcal{U}\left(\frac{5\pi}{4},\frac{\pi}{2},\frac{13\pi}{4}\right)$ and $a^2=b^2=c^2=d^3=e^3=f^3=g^3=-\mathbb{I}$. We have also divided the real number part of each group element by 2π. The class $C_0(n)$ describes defects in which the phase of the spinor is changed by $2\pi n$ as the defect line is encircled. Note that only $C_0(0)$ corresponds to trivial defects. In the case of $\overline{C_0}(n)$ phase change of $2\pi n$ is accompanied by a 360° rotation about z-axis. For the rest of the conjugacy classes an explicit description of the defects is more complicated. For example, the element (n,a) in the class $C_2(n)$ depicts a defect in which the spinor rotates through 180° about the z-axis and changes phase by $2\pi n$ as the line is encircled. Similarly $(n,b)\in C_2(n)$ describes rotations first through 90° about the z-axis and then through 180° about the y-axis together with $2\pi n$ phase change. However, because these defects belong to the same conjugacy class they can be continuously transformed into one another.

The multiplication table of conjugacy classes is shown in table 4. It shows that, for example, when we combine defect $C_2(n)$ with $C_2(-n)$ they can either annihilate each other ($C_0(0)$) or form defect $\overline{C_0}(0)$ or $C_2(0)$, the result depending on how they are brought together.

Defects can be classified further using homology groups [16, 17]. In the presence of other line singularities it may be possible to transform two line defects described by different conjugacy classes into one another. This is achieved by splitting a defect into two parts and combining these beyond a suitable line defect. Elements of $\pi_1(M)$ can be grouped into sets in such a way that defects described by elements in the same set can be deformed into one another either continuously or in the previously described way. The collection of these sets forms a factor group $\pi_1(M)/D$, where D is an invariant subgroup of $\pi_1(M)$ generated by elements $\delta\tau\delta^{-1}\tau^{-1}$ with $\delta,\tau\in\pi_1(M)$. The elements of $\pi_1(M)/D$ are unions of conjugacy

Table 5. General forms of the ground state spinors of $F = 2$ condensate in the presence of an external magnetic field. Energy is degenerate with respect to angles θ and ϕ.

	ξ^T	$\mathcal{E}$
F_1	$e^{i\theta}(1, 0, 0, 0, 0)$	$4c - 2p$
F_2	$e^{i\theta}(0, 0, 0, 0, 1)$	$4c + 2p$
F_1'	$e^{i\theta}(0, 1, 0, 0, 0)$	$c - p$
F_2'	$e^{i\theta}(0, 0, 0, 1, 0)$	$c + p$
C	$\frac{1}{2}\left(e^{i\theta}(1 + \frac{p}{4c}), 0, e^{i\phi}\sqrt{2 - \frac{p^2}{8c^2}}, 0, e^{-i(\theta-2\phi)}(-1 + \frac{p}{4c})\right)$	$-\frac{p^2}{4c}$
P	$\frac{1}{\sqrt{2}}\left(e^{i\theta}\sqrt{1 + \frac{p}{4c-d}}, 0, 0, 0, e^{i\phi}\sqrt{1 - \frac{p}{4c-d}}\right)$	$d - \frac{p^2}{4c-d}$
P_1	$\frac{1}{\sqrt{2}}\left(0, e^{i\theta}\sqrt{1 + \frac{p}{2(c-d)}}, 0, e^{i\phi}\sqrt{1 - \frac{p}{2(c-d)}}, 0\right)$	$d - \frac{p^2}{4(c-d)}$
P_0	$e^{i\theta}(0, 0, 1, 0, 0)$	d

classes. In our case D is the union of the conjugacy classes with winding number zero, $D = C_0(0) \cup \overline{C}_0(0) \cup C_2(0)$, and

$$\pi_1(G/H^C)/D = \left\{C_0 \cup \overline{C}_0 \cup C_2, C_3 \cup \overline{C_3}, C_3^2 \cup \overline{C_3^2}\right\}. \tag{5}$$

Here we have omitted winding numbers, which are n, $1/3 + n$ and $2/3 + n$, respectively. We see that line defects with the same winding number can be deformed to one another either continuously or using a splitting and recombination process.

From the work of Poenaru and Toulouse [14] we know that when two line defects (described by $\delta, \tau \in \pi_1(M)$) cross each other they produce a new line defect connecting them. This defect is of the type $\delta\tau\delta^{-1}\tau^{-1}$. Clearly, if $\delta\tau\delta^{-1}\tau^{1} = 1$, line defects can pass through each other without the creation of a new singular defect. In our case defects that can be created by making two line defects cross are the trivial defect $C_0(0)$ and two non-trivial defects, namely $\overline{C_0}(0)$ and $C_2(0)$.

4.2. *Non-zero external field*

Ground states were calculated in [7, 8] and are shown in table 5. However, now it should be noted that in the cyclic phase the order-parameter space has a quite complicated structure [8]. Group $U(1) \times SO(2)$ can act transitively on this order-parameter space only if the external field is strong enough, and even then there may be states which are degenerate in energy but which cannot be obtained from the reference order-parameter shown in table 5 [8].

In the ferromagnetic phases and in the P_0 phase $G/H = U(1)$ and the first and second homotopy groups are $\mathbb{Z}$ and 0. In C, P and P_1 phases $G/H = U(1) \times U(1)$ and the homotopy groups are $\mathbb{Z} \times \mathbb{Z}$ and 0. Physically this means that we can have a vortex in each component of a spinor but only two of them can have independent winding numbers. In the C phase, if there are vortices with winding numbers m and n say, in the first and third components of the spinor, then there must also be a vortex in the fifth component of the spinor. However, its winding number is not free but equal to $2n - m$.

5. Discussions

In this paper, we have calculated the first and second homotopy groups of the order-parameter spaces of spinor condensates with $F = 1$ and $F = 2$. The elements of these groups correspond to topologically stable singular line and point defects. The order-parameter space is identified with the set of degenerate ground state spinors, and both non-zero and zero external magnetic field cases are discussed.

In $F = 1$ condensate there are two possible phases, ferromagnetic and antiferromagnetic. If external field is zero in the former there can be one topologically non-trivial line defect but no topologically non-trivial point defects. In the latter infinitely many line and point defects, labelled by integers, are possible.

In $F = 2$ condensate three different phases, ferromagnetic, polar and cyclic are possible. The ferromagnetic phase can be further divided into two phases labelled by $\|\langle \mathbf{F}\rangle_\xi\| = 1$ or 2. In zero field the former has similar defects to the ferromagnetic phase of $F = 1$ condensate and in the latter there can be three topologically non-trivial line defects but point defects are not stable.

In the absence of an external field the order-parameter space of the cyclic phase has a non-commuting first homotopy group. Topologically stable defects are classified by the conjugacy classes of this group and are those in which the spinor is suitably rotated and its phase changed by an integer multiple of $\pi/3$ as the defect line is encircled. Stable point defects are not possible. If an external magnetic field is applied the symmetry is reduced and non-commutativity of the first homotopy group is lost. It also turns out that in the zero field $U(1) \times SO(3)$ does not act transitively on the order-parameter space of the polar phase and thus the defect structure remains unsolved.

For $F = 1$ and $F = 2$ condensates, if the external field is non-zero and there is only one non-zero component in the spinor, a vortex with an arbitrary integer winding number is possible. If there are two or three non-zero components then a vortex in each component of the spinor is possible, but only two of these can have an independent winding number. In the presence of a magnetic field stable point defects cannot exist.

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Appendix

The 3×3 representation matrix corresponding to $\mathcal{U}(\alpha, \beta, \gamma)$ is

$$D^{(1)}(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i(\alpha+\gamma)}\cos^2\frac{\beta}{2} & -e^{-i\alpha}\frac{1}{\sqrt{2}}\sin\beta & e^{-i(\alpha-\gamma)}\sin^2\frac{\beta}{2} \\ e^{-i\gamma}\frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -e^{i\gamma}\frac{1}{\sqrt{2}}\sin\beta \\ e^{i(\alpha-\gamma)}\sin^2\frac{\beta}{2} & e^{i\alpha}\frac{1}{\sqrt{2}}\sin\beta & e^{i(\alpha+\gamma)}\cos^2\frac{\beta}{2} \end{pmatrix}. \tag{A.1}$$

The five-dimensional representation matrix is given by $D^{(2)}(\alpha, \beta, \gamma) = \exp(-i\alpha F_z)\exp(-i\beta F_y)\exp(-i\gamma F_z)$, where $\exp(-i\alpha F_z) = \mathrm{diag}(e^{-i2\alpha}, e^{-i\alpha}, 1, e^{i\alpha}, e^{i2\alpha})$ and

$$\exp(-i\beta F_y) = \begin{pmatrix} \cos^4\frac{\beta}{2} & -\sin\beta\cos^2\frac{\beta}{2} & \frac{\sqrt{6}}{4}\sin^2\beta & -\sin\beta\sin^2\frac{\beta}{2} & \sin^4\frac{\beta}{2} \\ \sin\beta\cos^2\frac{\beta}{2} & \frac{1}{2}(\cos\beta+\cos2\beta) & -\frac{\sqrt{6}}{4}\sin2\beta & \frac{1}{2}(\cos\beta-\cos2\beta) & -\sin\beta\sin^2\frac{\beta}{2} \\ \frac{\sqrt{6}}{4}\sin^2\beta & \frac{\sqrt{6}}{4}\sin2\beta & \frac{1}{4}(1+3\cos2\beta) & -\frac{\sqrt{6}}{4}\sin2\beta & \frac{\sqrt{6}}{4}\sin^2\beta \\ \sin\beta\sin^2\frac{\beta}{2} & \frac{1}{2}(\cos\beta-\cos2\beta) & \frac{\sqrt{6}}{4}\sin2\beta & \frac{1}{2}(\cos\beta+\cos2\beta) & -\sin\beta\cos^2\frac{\beta}{2} \\ \sin^4\frac{\beta}{2} & \sin\beta\sin^2\frac{\beta}{2} & \frac{\sqrt{6}}{4}\sin^2\beta & \sin\beta\cos^2\frac{\beta}{2} & \cos^4\frac{\beta}{2} \end{pmatrix}. \tag{A.2}$$

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